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Bayesian Estimation for Pareto Type II Distribution using Monte-Carlo Techniques based on Record Values

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ABSTRACT

The present paper is devoted to obtain the Bayes estimators of the unknown parameters of the Pareto Type II distribution under the assumptions of gamma priors on both the shape and scale parameters are considered. The Bayes estimators cannot be obtained in explicit forms. So we propose Markov Chain Monte Carlo (MCMC) techniques to generate samples from the posterior distributions and in turn computing the Bayes estimators. Point estimation and confidence intervals based on maximum likelihood is also proposed. The approximate Bayes estimators obtained under the assumptions of informative as well as non-informative priors, are compared with the maximum likelihood estimators using Monte Carlo simulations. One real data set has been analyzed for illustrative purposes.

Keywords: Pareto Type II Distribution, Record Values, Bayesian Estimation, Simulation and MCMC Techniques.

1 Introduction

In the present time, the theory of record values and its applications are widely used in data analysis, espically in the study of stock market for making predictions about the price of a stock which may be higher or lower then the prior one. Ahsanullah (1995) and Arnold et al. (2011) have provided extensive use of record values for various real life situations. Nigm and Hamdy (1987) have mentioned that the Pareto Type II Distribution is within the category of distributions with decreasing failure rates. It has been observed by Dupuis and Tsao (1998), Castillo and Hadi (1997), Chhetri et al. (2017), Aslam et al. (2020) and Kerbaa et al. (2023) etc. and many more have been observed that Pareto Type II Distribution is widely applicable in the fields of engineering, biology, medicine and others, and more over this distribution is quiet helpful for the purpose of modelling and analysis life time data. In order to compute the credible intervals of the unknown parameters of the Pareto Type II distribution

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under the upper record values, the study provides Bayes estimates via Markov Chain Monte Carlo (MCMC) approaches. The future scope of this study would be on distribution in terms of extention of parameters along with the extention of Monte-Carlo techniques like Hamiltonion Monte-Carlo (HMC), Transitional Markov Chain Monte Carlo (TMCMC) and Transdimensional Transformation Markov Chain Monte Carlo (TTMCMC) techniques etc. Let $X_1, X_2, X_3, ..., X_n$ be a series of independent random variables with cumulative distribution function $F(x)$ and probability density function $f(x)$.
If $Y_j \geq Y_{j-1}; j \geq 1$, then X_j is referred to as an upper record and is denoted by $X_{U(j)}$ in the set $Y_n = max(X_1, X_2, X_3, ..., X_n)$, where $n \geq 1$. Let $X_{U(1)}, X_{U(2)}, X_{U(3)}, ..., X_{U(n)}$ be the first upper record values of size n resulting from a series of independent and identically Pareto variables with the probability density function

$$f(x) = \gamma\lambda^{\gamma}(x+\lambda)^{-(\gamma+1)}; \qquad x \geq 0; \gamma, \lambda > 0 \tag{1.1}$$

And cumulative distribution functiion

$$F(x) = 1 - \lambda^{\gamma}(x+\lambda)^{-\gamma}; \qquad x \geq 0; \gamma, \lambda > 0 \tag{1.2}$$

Where λ is scale and γ is shape parameter.

2 Estimation of Parameters under Maximum Likelihood Estimation(MLE)

Suppose that $\underline{x} = x_{u(1)}, x_{u(2)}, ..., x_{u(n)}$ be the first upper record values of size n from Pareto Kind ll distribution. The likelihood function for observed record $\underline{x}$ given by,

$$l(\gamma, \lambda|\underline{x}) = f(x_{u(n)}) \prod_{i=1}^{n-1} \frac{f(x_{u(i)})}{1 - F(x_{u(i)})} \tag{2.1}$$

Where $f(.)$ and $F(.)$ are given, respectively by (1.1) and (1.2). Substituting $f(.)$ and $F(.)$ in equation (2.1), we get

$$l(\gamma, \lambda|\underline{x}) = \gamma^n \lambda^{\gamma}(x_{u(n)} + \lambda)^{-\gamma} \prod_{i=1}^{n} (x_{u(i)} + \beta)^{-1} \tag{2.2}$$

Log likelihood function may be then written as

$$L(\gamma, \lambda|\underline{x}) = logl(\gamma, \lambda|\underline{x})$$

$$i.e, \quad L(\gamma,\lambda|\underline{x}) = nlog\gamma + \gamma log\lambda - log\gamma(x_{u(n)}+\lambda) - \sum_{i=1}^{n} log(x_{u(i)}+\lambda) \quad (2.3)$$

Taking derivatives with respect to γ and λ of (2.3) and equating them to zero, we obtain the likelihood equations for γ and λ to be

$$\frac{\partial L(\gamma,\lambda|\underline{x})}{\partial\gamma} = \frac{n}{\gamma} + log\lambda - log(x_{u(n)}+\lambda) \quad (2.4)$$

$$\frac{\partial L(\gamma,\lambda|\underline{x})}{\partial\lambda} = \frac{\gamma}{\lambda} + \frac{\gamma}{(x_{u(n)}+\lambda)} - \sum_{i=1}^{n}\frac{1}{(x_{u(i)}+\lambda)} \quad (2.5)$$

The equations (2.4) and (2.5) cannot solve analytically for γ and λ. Therefore, we use R software to solve these equations and find the MLE's of the unknown parameters γ and λ.
The asymptotic variances and covariances of the MLE for parameters γ and λ are given by elements of the inverse of the Fisher information matrix given by

$$\mathbf{I}_{ij} = E\left[-\frac{\delta^2 L}{\delta\gamma\delta\lambda}\right]; \qquad i,j=1,2 \quad (2.6)$$

Deleting the expectation operator E, we obtain the estimated asymptotic variance-covariance matrix for the MLE, as

$$\begin{bmatrix} -\frac{\delta^2 L(\gamma\lambda|\underline{x})}{\delta\gamma^2} & -\frac{\delta^2 L(\gamma\lambda|\underline{x})}{\delta\gamma\delta\lambda} \\ -\frac{\delta^2 L(\gamma\lambda|\underline{x})}{\delta\lambda\delta\gamma} & -\frac{\delta^2 L(\gamma\lambda|\underline{x})}{\delta\lambda^2} \end{bmatrix}^{-1}_{(\hat{\gamma},\hat{\lambda})} = \begin{bmatrix} cov(\hat{\gamma}) & cov(\hat{\gamma},\hat{\lambda}) \\ cov(\hat{\lambda},\hat{\gamma}) & cov(\hat{\lambda}) \end{bmatrix},$$

With

$$\frac{\delta^2 L(\gamma\lambda|\underline{x})}{\delta\gamma^2} = -\frac{n}{\gamma^2} \quad (2.7)$$

$$\frac{\delta^2 L(\gamma,\lambda|\underline{x})}{\delta\gamma\delta\lambda} = \frac{\delta^2 L(\gamma,\lambda|\underline{x})}{\delta\lambda\delta\gamma} = \frac{1}{\lambda} - \frac{1}{(x_{u(n)}+\lambda)} \quad (2.8)$$

$$\frac{\delta^2 L(\gamma,\lambda|\underline{x})}{\delta\lambda^2} = \frac{-\gamma}{\lambda^2} - \frac{1}{(x_{u(n)}+\lambda)^2} + \sum_{i=1}^{n}\frac{1}{(x_{u(n)}+\lambda)^2} \quad (2.9)$$

The $(1-\alpha)100\%$ confidence intervals for parameter γ and λ given by

$$\hat{\gamma} \pm Z_{\alpha/2}\sqrt{var(\hat{\gamma})} \ \ and \ \ \hat{\lambda} \pm Z_{\alpha/2}\sqrt{var(\hat{\lambda})} \quad (2.10)$$

where $Z_{\alpha/2}$ is the percentile of standard normal distribution with right-tail probability $\alpha/2$.

3 Estimation of parameters under MCMC

Here, We discuss Bayesian method and MCMC algorithm for computation. The posterior samples are generated using Metropolis Hastings method. The Metropolis-Hastings algorithm is one of the most popular MCMC algorithm. Like other MCMC methods, the Metropolis-Hastings algorithm is used to generate serially correlated draws from a sequence of probability distributions. The sequence converges to a given target distribution. which are then utilised to compute the Bayes point estimates and construct the relevant credible intervals based on the posterior samples.
Assume the following gamma prior densities for model (1.1).

$$\pi_1(\gamma|p,q) = \begin{cases} \frac{q^p}{\Gamma p}\gamma^{p-1}exp(-q\gamma) & (\gamma \geq 0) \\ 0 & (\gamma < 0) \end{cases} \tag{3.1}$$

$$\pi_1(\lambda|r,s) = \begin{cases} \frac{s^r}{\Gamma r}\lambda^{r-1}exp(-s\lambda) & (\lambda \geq 0) \\ 0 & (\lambda < 0) \end{cases} \tag{3.2}$$

The joint prior density of γ and λ may be written as

$$\pi(\gamma,\lambda) = \pi_1(\gamma|p,q)\pi_2(\lambda|r,s)$$
$$= \frac{q^p s^r}{\Gamma p \Gamma r}\gamma^{p-1}\lambda^{r-1}exp(-q\gamma - s\lambda) \tag{3.3}$$

The joint posterior density of γ and λ is

$$\pi^*(\gamma,\lambda|\underline{x}) = \frac{l(\gamma,\lambda|\underline{x})\pi(\gamma,\lambda)}{\int_0^\infty \int_0^\infty l(\gamma,\lambda|\underline{x})\pi(\gamma,\lambda)d\gamma d\lambda} \tag{3.4}$$

Therefore, the Bayes estimate of any function of γ and λ say (γ,λ), under square error loss function is

$$\tilde{g}(\gamma,\lambda) = E_{\gamma,\lambda|data}[g(\gamma,\lambda)] = \frac{\int_0^\infty \int_0^\infty g(\gamma,\lambda)l(\gamma,\lambda|\underline{x})\pi(\gamma,\lambda)d\gamma d\lambda}{\int_0^\infty \int_0^\infty l(\gamma,\lambda|\underline{x})\pi(\gamma,\lambda)d\gamma d\lambda} \tag{3.5}$$

Now, we utilise the MCMC technique to produce samples from the posterior distribution and then compute the Bayes Estimator of $g(\gamma,\lambda)$ under the squared error loss(SEL) function. The ratio of two integrals provided by (3.5) cannot be achieved in a closed form, therefore we use MCMC methods for approximation. See for example Robert et al. (2010)

MCMC Approach

There are many different MCMC schemes available, and selecting one might be challenging. Gibbs sampling and the broader Metropolis-within-Gibbs samplers are crucial MCMC subclasses. The benefit of adopting the MCMC technique over the MLE method is that by building the probability intervals based on the empirical posterior distribution, we can always get an acceptable interval estimate of the parameters. In fact, using a kernel estimate of the posterior distribution, the MCMC samples may be used to fully summarise the posterior uncertainty regarding the parameters and any function of the arguments has the same properties. By multiplying the likelihood by the joint prior, the equation for the joint posterior up to proportionality may be desired as

$$\pi^*(\gamma,\lambda) \propto \gamma^{n+p-1}\lambda^{n+r-1} exp[-(q\gamma + d\lambda - \lambda log(1-exp(-x^{\gamma}_{u(n)})))] + \prod_{i=1}^{n-1} \frac{{x_{u(i)}}^{\gamma-1} exp(-x^{\gamma}_{u(i)})}{1-exp(-x^{\gamma}_{u(i)})} \tag{3.6}$$

The posterior distribution of the supplied Eq. (3.6) cannot be analytically reduced to well-known distributions, making it impossible to sample directly using conventional methods. However, the plot of the posterior distribution indicates that it is comparable to the normal distribution. We thus employ the Metropolis-Hastings technique with the normal proposal distribution to get random integers from this distribution. The selection of the hyperparameters $(p, q, r\ and\ s)$ that brings (3.6) close to the proposal distribution and definitely increases the MCMC iteration's convergence. To select samples from the posterior density functions, we compute the Bayes estimates, and also create the associated credible interval.

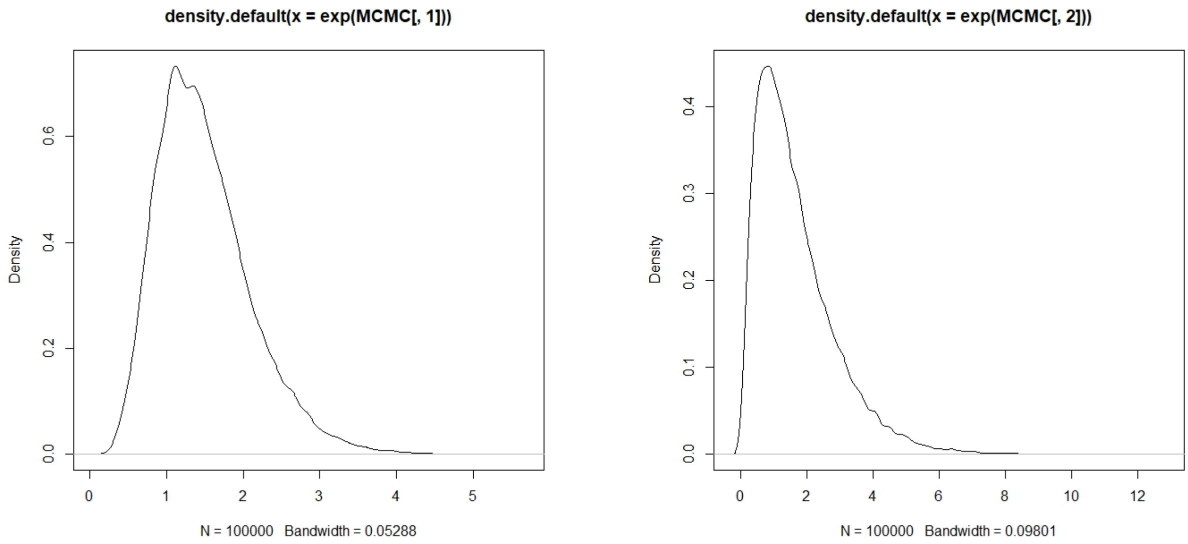
4 Application to Real Data

Interpreting an application to real data involves the process of analyzing and making sense of the results or output generated by a specific application or model when it is applied to real-world data. This interpretation is crucial for understanding the significance, implications, and limitations of the application. We selected actual data that Choulakian and Stephens (2001) had also utilised. The data represent the Wheaton River in Carcross, Yukon Territory, Canada, exceedances of flood maxima $(in\ m^3/s)$. The statistics are excesses for the years 1958 through 1984. The data are given below

1.7,2.2,14.4,1.1,0.4,20.6,5.3,0.7,13.0,12.0,9.3,1.4,
18.7,8.5,25.5,11.6,2.2,39.0,0.3,15.0,14.1,22.1,1.1,2.5,

14.4,1.7,37.6,0.6,11.0,7.3,22.9,1.7,0.1,1.1,0.6,9.0,
1.7,7.0,20.1,0.4,14.1,9.9,10.4,10.7,30.0,3.6,5.6,30.8,
13.3,4.2,25.5,3.4,11.9,21.5,27.6,36.4,2.7,64.0,1.5,2.5,
27.4,1.0,27.1 ,20.2,16.8,5.3,9.7,7.5,2.5,27.0,1.9,2.8

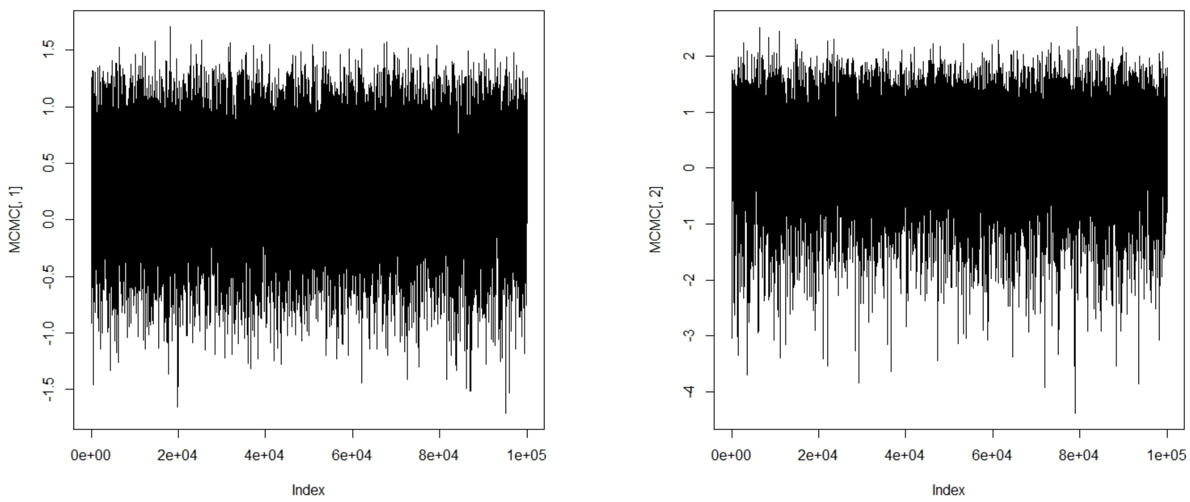
As a result, we see the following upper record values in the observed data: $1.7, 2.2, 14.4, 20.6, 39, 64$ Based on these seven upper record values, we compute the approximate MLEs and Bayes estimates of γ and λ using MCMC method. We use less informative and informative prior on both γ and λ. Also 95% approximate MLE confidence intervals and approximate credible intervals based on MCMC samples, the results are given in table 1. We have done with 1000000 MCMC samples.



(a) density plot of γ (b) density plot of λ

Figure 1: Density plot of γ and λ

The density plot of γ provides information about the likelihood or frequency of different values of γ. It helps in understanding the distribution of the parameter γ and its shape. Similarly, the density plot of λ provides information about the likelihood or frequency of different values of λ. It helps in understanding the distribution of the parameter λ and its shape. Both density plots give insights into the distribution of the parameters γ and λ, which are important for analyzing and interpreting the data.



(a) Trace plot of γ (b) Trace plot of λ

Figure 2: Trace plot of γ and λ

The Figure 2 is the trace plot of the parameter γ and λ. A trace plot is a graphical representation that shows the values of a parameter over iterations in a Markov Chain Monte Carlo (MCMC) simulation. In other words we can say it provides a visual representation of how the parameters are changing throughout the sampling process. In these both cases, the trace plot of γ indicates how the value of γ changes over iterations and the trace plot of λ indicates how the value of λ changes over iterations in the MCMC simulation. It provides insights into the convergence and stability of the MCMC algorithm for estimating the parameters γ and λ.

Table 1: Estimates of γ and λ obtained by MLE and MCMC

Method	Parameter	Point	Interval	Length
MLEs	γ	2.3935	[1.6487,3.1683]	1.5496
	λ	5.6822	[3.3026,8.0618]	4.7592
MCMC(1)	γ	1.5070	[-0.4892,1.0087]	1.4979
	λ	1.6915	[-0.7910,1.5149]	2.3059
MCMC(2)	γ	1.3212	[-0.3695,1.0052]	1.3747
	λ	1.4259	[-0.5689,1.1012]	1.6701

Table 1 provides the approximate maximum likelihood estimates (MLEs) and Bayes estimates of the parameters γ and λ using the Markov Chain Monte Carlo (MCMC) method. It also includes the 95 percent approximate MLE confidence intervals and approximate credible intervals based on the MCMC samples. The results in Table 1 are obtained using both less informative and informative priors on γ and λ, with 1,000,000 MCMC samples.

Simulation Study

We created simulated upper record samples from a Pareto Kind ll distribution to assess how the suggested approaches behaved. To compare the MLEs and other Bayes estimators as well as to investigate their influence on various parameter values using various sample sizes (n), different hyperparameters (p, q, r, s) and three sets of parameter values $(\gamma, \lambda) = (2, 3)(2, 2)(2.5, 2.5)$. Using informative prior $p = 4, q = 2, r = 0.4$ and $s = 0.2$ together with the less informative gamma priors for both parameters. In order to calculate the Bayes estimates, we employed the squared error loss function. Finally, calculate various estimations. Results from MLEs and Bayes estimators with informative (*in tables written as* $MCMC(2)$) and less informative priors (*in tables written as* $MCMC(1)$) on both and are presented in Tables $2-4$.

Table 2: Different Estimators and risk of corresponding estimators when (γ, λ) = (2, 3)

n	MLE		MCMC(1)		MCMC(2)	
	γ	λ	γ	λ	γ	λ
5	0.7741 ***(76.73)***	0.4248 ***(79.75)***	1.0566 ***(74.47)***	1.2564 ***(72.97)***	0.8869 ***(75.81)***	0.8806 ***(75.86)***
15	0.8964 ***(38.34)***	0.2651 ***(41.73)***	1.2355 ***(36.85)***	1.1249 ***(37.31)***	0.8867 ***(38.39)***	0.8767 ***(38.43)***
25	1.0860 ***(60.76)***	1.0088 ***(61.12)***	1.1329 ***(60.55)***	1.2862 ***(59.88)***	0.8894 ***(61.71)***	0.8786 ***(61.76)***
35	1.4949 ***(266.39)***	3.5038 ***(255.63)***	1.1886 ***(268.73)***	1.6401 ***(265.34)***	0.8857 ***(271.25)***	0.8771 ***(271.32)***
50	1.4962 ***(82.33)***	0.5038 ***(88.54)***	1.5674 ***(81.96)***	0.9189 ***(85.70)***	0.8941 ***(85.86)***	0.8220 ***(86.33)***

Note: The first figure represents estimates with the corresponding average expected loss over sample space (risk of corresponding estimators reported below it in parentheses).

Table 2, table 3 and table 4 presents different estimators and the corresponding risk of the estimators when the parameters (γ, λ) are set to $(2, 3)$, $(2, 2)$ and $(2.5, 2.5)$. These estimators include the maximum likelihood estimator (MLE), the Bayes estimator with a less informative prior ($MCMC(1)$), and the Bayes estimator with an informative prior ($MCMC(2)$). The values in the tables represent the estimates of γ and λ, along with the average expected loss over the sample space (risk of the estimators). From the tables, we can observe that the MLEs and the Bayes estimators based on both the less informative and informative priors provide estimates for γ and λ. The risk of the estimators varies for different sample sizes n ranging from 5 to 50. Overall, the tables provides a comparison of the performance of the different estimators in terms of their risk corresponding to the estimators.

Table 3: Different Estimators and risk of corresponding estimators when (γ, λ) = (2, 2)

n	MLE		MCMC(1)		MCMC(2)	
	γ	λ	γ	λ	γ	λ
5	0.6203 ***(7.882)***	0.0547 ***(9.473)***	1.2338 ***(6.892)***	0.8473 ***(7.423)***	0.8970 ***(7.336)***	0.8926 ***(7.344)***
15	1.3151 ***(10.67)***	0.6229 ***(11.95)***	1.4345 ***(10.55)***	1.0811 ***(11.00)***	0.8779 ***(11.37)***	0.8673 ***(11.39)***
25	1.7149 ***(5.729)***	1.2846 ***(6.021)***	1.5082 ***(5.823)***	1.2616 ***(8.202)***	0.8729 ***(6.647)***	0.8673 ***(6.642)***
35	0.7384 ***(163.25)***	1.0146 ***(161.25)***	0.8495 ***(162.43)***	1.3353 ***(159.12)***	0.8873 ***(162.16)***	0.8825 ***(162.19)***
50	0.7474 ***(161.57)***	0.4180 ***(163.37)***	0.8831 ***(160.88)***	0.8757 ***(160.92)***	0.8801 ***(160.90)***	0.8755 ***(160.92)***

Note: The first figure represents estimates with the corresponding average expected loss over sample space (risk of corresponding estimators reported below it in parentheses).

Table 4: Different Estimators and risk of corresponding estimators when (γ, λ) = (2.5, 2.5)

n	MLE		MCMC(1)		MCMC(2)	
	γ	λ	γ	λ	γ	λ
5	0.6577 ***(8.663)***	0.3617 ***(9.408)***	1.0486 ***(7.948)***	1.2613 ***(7.683)***	0.8979 ***(8.187)***	0.8927 ***(8.197)***
15	0.6577 ***(7.153)***	0.3617 ***(8.069)***	1.7636 ***(5.278)***	0.7244 ***(6.970)***	0.8846 ***(6.568)***	0.8749 ***(6.591)***
25	2.3810 ***(17.76)***	8.6560 ***(58.93)***	1.1144 ***(19.01)***	1.6414 ***(18.10)***	0.8994 ***(19.53)***	0.8923 ***(19.55)***
35	2.2582 ***(3.5699)***	1.8489 ***(3.1953)***	1.6395 ***(3.1331)***	1.2828 ***(3.2291)***	0.8738 ***(3.6524)***	0.8671 ***(3.6622)***
50	2.0359 ***(9.2462)***	2.3450 ***(9.3989)***	1.4638 ***(9.4678)***	1.3616 ***(9.5763)***	0.8889 ***(10.3498)***	0.8708 ***(10.3883)***

Note: The first figure represents estimates with the corresponding average expected loss over sample space (risk of corresponding estimators reported below it in parentheses).

Conclusion

In this study, we address the Bayes estimate of the Pareto Type II distribution's unknown parameters when the data are higher record values. Under the suppositions of squared error loss function, we present the Bayes estimators and assume the gamma priors on the unknown parameters. The Bayes estimators can be produced via numerical integration, however they cannot be obtained in explicit forms. Because of this, we generated posterior sample using the MCMC approach. We observe the following

1. From the results obtained in Tables $2-4$. It can be seen that the performance of the Bayes estimators with respect to the less-informative prior is quite close to that of the MLEs.
2. Tables $2-4$ report the results based on less-informative prior and informative prior, also in these case the results based on using the MHA are quite similar in nature when comparing the Bayes estimators based on informative prior clearly shows that the Bayes estimators based on informative prior perform better than the MLEs, in terms of the risk corresponding estimator.
3. From Tables $2-4$, it is clear that the Bayes estimators based on informative prior perform much better than less-informative prior and the MLEs in terms of the risk corresponding estimator.

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7 Declarations of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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